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Cylindrically symmetric charged perfect-fluid distribution in general relativity

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In the present paper a solution of Einstein-Maxwell field equations in cylindrically symmetric space-time is derived for the perfect fluid case. Expressions for material density and charge density have been explicitly found.

1. Introduction

The solutions of Einstein field equations have been obtained by several authors either in purely gravitational case or in purely electromagnetic case. The problem of finding the solution when both the fields are superposed is of considerable physical interest. Recently Shah & Vaidya (1967) have considered the superposition of these two fields and found a solution in the case of spherically symmetric space-time. Herein we investigate a cylindrically symmetric solution of Einstein=Maxwell field equations corresponding to charged perfect fluid distribution.

2. Field Equations

Choosing co-moving coordinates (ρ, ϕ, z) the line element exhibiting cylindrical symmetry given by Marder (1958) can be put in the form :

$$ds^2 \exp\{2(\alpha-\beta\}(dt^2-d\rho^2)-\rho^2\exp\{(-2\beta)\}d\phi^2-\exp\{2(\beta+\gamma)\}dz^2 \tag{2.1}$$

where α, β, γ are arbitrary functions of ρ and t only.

The field equations

$$R_i^k-\tfrac{1}{2}R\delta_i^k = -8\pi T_i^k. \tag{2.2}$$

take the form

$$-8\pi T_1^1 = \exp\{-2(\alpha-\beta)\} \times \left(\ddot{\alpha}\gamma-2\dot{\beta}\dot{\gamma}-\beta'^2-2\beta'\gamma'+\alpha'\gamma'+\frac{\alpha'}{\rho}+\frac{\gamma'}{\rho}-\ddot{\gamma}-\dot{\gamma}^2-\dot{\beta}^2\right) \tag{2.3}$$

$$-8\pi T_2^2 = \exp\{-2(\alpha-\beta)\}\times (-2\dot{\beta}\dot{\gamma}+2\beta'\gamma'+\alpha''-\ddot{\alpha}+\beta'^2+\gamma''+\gamma'^2-\ddot{\gamma}-\dot{\gamma}^2-\dot{\beta}^2) \quad \ldots \tag{2.4}$$

$$-8\pi T_3^3 = \exp\{-2(\alpha-\beta)\}\times$$

$$\left(-2\beta''+2\ddot{\beta}-\frac{2\beta'}{\rho}+\alpha''-\ddot{\alpha}+\beta'^2-\dot{\beta}^2\right) \quad \ldots \quad (2.5)$$

$$-8\pi T_4^4 = \exp\{-2(\alpha-\beta)\}\times$$

$$\left(-\alpha'\gamma'+2\beta'\gamma'+\dot{\beta}^2+2\dot{\beta}\dot{\gamma}-\dot{\alpha}\dot{\gamma}-\frac{\alpha'}{\rho}+\beta'^2+\gamma''+\gamma'^2+\frac{\gamma'}{\rho}\right) \quad \ldots \quad (2.6)$$

$$-8\pi T_4^1 = 8\pi T_1^4 = -\exp\{-2(\alpha-\beta)\}\times$$

$$\left(\dot{\gamma}'+\gamma'\dot{\gamma}+2\beta'\dot{\beta}+2\beta'\dot{\gamma}+2\gamma'\dot{\beta}-\dot{\alpha}\gamma'-\alpha'\gamma-\frac{\dot{\alpha}}{\rho}\right) \quad \ldots \quad (2.7)$$

Here and in what follows, primes indicate differentiations with regard to ρ and overhead dots denote differentiations with regard to t

For a distribution of charged fluid we take

$$T_i^k = M_i^k+E_i^k \quad (2.8)$$

with

$$M_i^k = (p^*+\rho^*)V_iV^k-p^*\delta_i^k \quad (2.9)$$

$$V^1 = V^2 = V^3 = 0, \quad V_iV^i = 1 \quad (2.10)$$

and

$$4\pi E_i^k = -F^{ka}F_{ia}+\tfrac{1}{4}\delta_i^kF^{ab}F_{ab} \quad (2.11)$$

with

$$F[ij,k] = 0 \quad (2.12)$$

and

$$F^{ik};k = 4\pi L^i. \quad (2.13)$$

For the sake of simplicity we consider F_{14} as the only surviving component of F_{ik} Also, since we are using comoving co-ordinates the charge current vector L^i will have components (0, 0, 0, L^4). With the form of T_i^k given by (2.8) above, we find that

$$T_1^1 = -p^*-\frac{1}{8\pi}F^{14}F_{14} \quad \ldots \quad (2.14)$$

$$T_2^2 = T_3^3 = -p^*+\frac{1}{8\pi}F^{14}F_{14} \quad \ldots \quad (2.15)$$

$$T_4^4 = \rho^*-\frac{1}{8\pi}F^{14}F_{14} \quad \ldots \quad (2.16)$$

$$T_4^1 = T_1^4 = 0. \quad \ldots \quad (2.17)$$

3. Solution of the Field Equations

Substituting the values of T_2^2 and T_3^3 from (2.4) and (2.5) in (2.15), we get

$$-2\dot{\beta}\dot{\gamma}+2\beta'\gamma'+\gamma''+\gamma'^2-\ddot{\gamma}-\dot{\gamma}^2+2\beta''-2\ddot{\beta}+\frac{2\beta'}{\rho}=0 \quad \ldots \quad (3.1)$$

Thus we have two equations (2 17) and (3.1) to determine three unknowns α, β and γ The third equation is supplied by the equation of state of the fluid. We replace the equation of state by the assumption,

$$\gamma = \text{constant} \quad \ldots \quad (3.2)$$

Equations (2.17) and (3.1) give

$$2\beta'\dot{\beta}-\frac{\dot{\alpha}}{\rho}=0 \quad \ldots \quad (3.3)$$

and

$$\beta''-\ddot{\beta}+\frac{\beta'}{\rho}=0 \quad \ldots \quad (3.4)$$

respectively.

The solution of equation (3.4) is (Coulson 1955)

$$\beta = J_0(p\rho)\{a \cos pt+b \sin pt\} \quad \ldots \quad (3.5)$$

where a, b and p are arbitrary constants and

$$J_0(p\rho)=\sum_{r=0}^{\infty}\frac{(-1)^r\left(\frac{p\rho}{2}\right)^{2r}}{\lfloor r\,\Gamma r+1} \quad \ldots \quad (3.6)$$

is the Bessel function of order zero.

Substituting the value of β from (3.5) in (3.3) and then solving we get

$$\alpha = \rho J_0(p\rho)J'_0(p\rho)[c \sin 2pt-d \cos 2pt+f(\rho)] \quad \ldots \quad (3.7)$$

where for the sake of simplicity we write

$$J'_0(p\rho) \text{ for } \frac{\partial}{\partial\rho}J_0(p\rho), \quad c=ab, \quad 2d=b^2-a^2$$

and f is some function of ρ

Thus the solution of the field equations is given by the line element

$$\begin{aligned} ds^2 = {} & \exp[2J_0\{\rho J'_0(c \sin 2pt-d \cos 2pt+f(\rho))-(a \cos pt+b \sin pt)\}][dt^2-dp^2]- \\ & -\rho^2\exp[-2J_0(a \cos pt+b \sin pt)]\, d\phi^2- \\ & -\exp[2\{J_0(a \cos pt+b \sin pt)+K\}].dz^2. \end{aligned} \quad \ldots \quad (3.8)$$

Using equations (2.2), (2.14), (2.15) and (2.16) we find that

$$8\pi p^* = \tfrac{1}{2}\exp[-2J_0\{\rho J'_0(c\sin 2pt-d\cos 2pt+f(\rho))-(a\cos pt+b\sin pt)\}]\times$$
$$\times\left\{\left(\frac{J_0J'_0}{\rho}+3J_0'^2+3J_0J_0''+3\rho J'_0J''_0+\rho J_0J_0'''+4p^2\rho J_0J'_0\right)\times\right.$$
$$\times(c\sin 2pt-d\cos 2pt)+J_0J'_0\frac{f}{\rho}+3J_0'^2f+3J_0J''_0f+$$
$$+3J_0J'_0f'+3\rho J'_0J''_0f+2\rho J_0'^2f'+\rho J_0J_0'''f+2\rho J_0J''_0f'+$$
$$\left.+\rho J_0J'_0f''-2J_0^2p^2(-a\sin pt+b\cos pt)^2\right\} \quad \ldots (3.9)$$

$$8\pi\rho^* = \tfrac{1}{2}\exp[-2J_0\{\rho J'_0(c\sin 2pt-d\cos 2pt+f(\rho))-(a\cos pt+b\sin pt)\}]\times$$
$$\left\{\left(3\frac{J_0J'_0}{\rho}+J_0'^2+J_0J_0''-3\rho J_0'J_0''-\rho J_0J_0'''-4p^2\rho J_0J_0'\right)\times\right.$$
$$(c\sin 2pt-d\cos 2pt)+\frac{3}{\rho}J_0J_0'f+J_0'^2f+J_0J_0''f+J_0J_0'f'-$$
$$-3\rho J_0'J_0''f-2\rho J_0'^2f'-\rho J_0J_0'''f-2\rho J_0J_0''f'-\rho J_0J_0'f''-$$
$$\left.-2J_0^2p^2(-a\sin pt+b\cos pt)^2-4J_0'^2(a\cos pt+b\sin pt)^2\right\} \quad \ldots (3.10)$$

$$F^{14} = \frac{1}{\sqrt{2}}\exp[-3J_0\{\rho J_0'(c\sin 2pt-d\cos 2pt+f(\rho))-(a\cos pt+b\sin pt)\}]$$
$$\left[\left\{J_0'^2+J_0J_0''+3\rho J_0'J_0''+\rho J_0J_0'''+4p^2\rho J_0J_0'-J_0\frac{J_0'}{\rho}\right\}\times\right.$$
$$(c\sin 2pt-d\cos 2pt)+J_0'^2f+J_0J_0''f+J_0J_0'f'+3\rho J_0'J_0''f$$
$$+2\rho J_0'^2f'+\rho J_0J_0'''f+2\rho J_0J_0''f'+\rho J_0J_0'f''+$$
$$\left.+2J_0'^2(a\cos pt+b\sin pt)^2-J_0J_0'\frac{f}{\rho}\right]^{\frac{1}{2}} \quad \ldots (3.11)$$

The charge density σ is defined by $\sigma = S\,|\{L^a L_a\}^{\frac{1}{2}}|$ where S is the sign of L^4 Therefore

$$4\pi\sigma = \exp[-2J_0\{\rho J_0'(c\sin 2pt-d\cos 2pt+f(\rho))-(a\cos pt+b\sin pt)\}]$$
$$\left[\frac{3\sqrt{q}}{\sqrt{2}}\{(c\sin 2pt-d\cos 2pt+f(\rho))(J_0J_0'+\rho J_0'^2+\rho J_0J_0'')+\right.$$
$$+\rho J_0J_0'f'\}-\frac{q'}{2\sqrt{2q}}-\frac{2\sqrt{q}}{\sqrt{2}}\left\{J_0J_0'+\rho J_0'^2+\right.$$
$$+\rho J_0J_0''(c\sin 2pt-d\cos 2pt)+J_0J_0'f+\rho J_0'^2f$$
$$\left.\left.+\rho J_0J_0''f+\rho J_0J_0'f'-J_0'(a\cos pt+b\sin pt)+\frac{1}{2\rho}\right\}\right] \quad \ldots (3.12)$$

where

$$q = \left\{J_0'^2+J_0J_0''+3\rho J_0'J_0''+\rho J_0J_0'''+4p^2\rho J_0J_0'-\frac{J_0J_0'}{\rho}\right\}$$

$$\times(c \sin 2pt-d \cos 2pt)+J_0'^2f+J_0J_0''f+J_0J_0'f'+$$

$$+3\rho J_0'J_0''f+2\rho J_0'^2f'+\rho J_0J_0'''f+2\rho J_0J_0''f'+$$

$$+\rho J_0J_0'f''+2J_0'^2(a \cos pt+b \sin pt)^2-J_0J_0'\frac{f}{\rho} \tag{3.13}$$

Thus from (3 12) and (3.10) the ratio of the charge and mass densities is obtained.

References

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